

Multipartite positive-partial-transpose inequalities exponentially stronger than local reality inequalities

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We show that positivity of *every* partial transpose of N -partite quantum states implies new inequalities on Bell correlations which are stronger than standard Bell inequalities by a factor of $2^{(N-1)/2}$. A violation of the inequality implies the system is in a bipartite distillable entangled state. It turns out that a family of N -qubit bound entangled states proposed by Dür [Phys. Rev. Lett. **87**, 230402 (2001)] violates the inequality for $N \geq 4$.

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I. INTRODUCTION

The striking feature of quantum mechanics is the existence of entangled states. There is much research of nature of entangled states related to local realistic theories [1, 2, 3]. Separable state is defined by Werner in 1989 [4]. And what state is separable or entangled was discussed very much [5, 6, 7, 8].

In 1996, Peres-Horodecki opened the method how to classify a state in question. A multipartite state ρ has positive partial transposes with respect to all subsystems if ρ is separable [9]. However it was shown [10] that this criterion is not sufficient to be separable in general. Such an entangled state was called as bound entangled states.

It was shown that all quantum states with positive partial transposes with respect to all subsystems satisfy Bell inequality with an arbitrary number of settings per site [11]. This result further supports the conjecture by Peres, that all such states can admit local realistic theories [12].

The enormous research of quantum information theory [13, 14] relies on utilizing entangled pure state. Therefore, it was discussed how to distill [15, 16] entangled pure states from a set of many entangled states with the presence of experimental noise, i.e., from a set of entangled mixed states. In the bipartite case, one cannot distill bipartite entangled pure state from a set of bipartite bound entangled states because only local operations and classical communication (LOCC) cannot change the property of positivity of partial transpose which a quantum state in question has.

In the multipartite case, the situation becomes complicated. In 2001, Dür proposed a family of N -qubit bound entangled states [17]. It was shown that those states violate Bell-Mermin inequality [18] for $N \geq 8$. Dür took the definition of the bound entangled states, as the ones from which one cannot distill a single copy of some entangled pure state from a set of multipartite bound entangled states. However, Acín showed that a violation of the Bell inequality reveals bipartite distillable entanglement [19]. That is, there is at least one bipartite splitting of the system such that the state becomes distillable, i.e., one can distill a single copy of bipartite entangled pure state from a set of multipartite bound entangled states.

Therefore, Dür bound entangled states are bipartite distillable entangled states for $N \geq 8$. After that, it was shown that Dür bound entangled states violate Bell inequality [20] with three settings per site for $N \geq 7$ [21]. Just after that, it turned out that Dür bound entangled states violate Bell inequality [22] with a continuous range of settings of the apparatus at each site for $N \geq 6$ [23].

And recently, it was shown [24] that separability of N -partite quantum states implies new inequalities on Bell correlations which are stronger than standard Bell inequalities [18, 25, 26] by a factor of $2^{(N-1)/2}$. So far, the relation between positivity of partial transpose of quantum states and Bell-Mermin inequality was researched by Werner and Wolf [27]. However, it has not yet reported our knowledge between the optimal upper bound of Bell-Mermin inequality and positivity of partial transpose of quantum states in detail.

In this paper, we show that positivity of *every* partial transpose of N -qubit quantum states implies new inequalities on Bell correlations which are stronger than standard Bell inequalities by a factor of $2^{(N-1)/2}$. We may call it ‘positive partial transpose inequalities’. One can also see that a violation of the new inequality which is tested by many parties implies that the system is in bipartite distillable entangled state since there is at least one bipartite splitting of the system such that the state becomes distillable as discussed in Ref. [19]. It turns out that Dür bound entangled states violate the new inequality for $N \geq 4$. Thus, a family of Dür bound entangled states has a property that they are bipartite distillable entangled states for $N \geq 4$.

This paper is organized as follows. In Sec. II, we derive a positive partial transpose inequality. And we shall show the new inequality is stronger than standard Bell inequalities by a factor of $2^{(N-1)/2}$. In Sec. III, it will be shown that Dür bound entangled states violate the new inequality for $N \geq 4$. One will see that a family of Dür bound entangled states has a property that they are bipartite distillable entangled states for $N \geq 4$. A short summary follows in Sec. IV.

II. POSITIVE PARTIAL TRANSPOSE INEQUALITY

In this section, we briefly review partial transpositions, next present a positive partial transpose inequality. If the partial transposes with respect to all subsystems are positive, the inequality is satisfied. And finally we show that positivity of *every* partial transpose of N -qubit quantum states implies new inequalities on Bell correlations which are stronger than standard Bell inequalities by a factor of $2^{(N-1)/2}$.

The partial transpose of an operator on a Hilbert space $H_1 \otimes H_2$ is defined by:

$$\left(\sum_l A_l^1 \otimes A_l^2 \right)^{T_1} = \sum_l A_l^{1T} \otimes A_l^2, \quad (1)$$

where the superscript T denotes transposition in the given basis. The positivity of partial transpose is found to be a necessary condition for separability [9, 10]. The operator obtained by the partial transpose of any separable state is positive (PPT - positive partial transpose). In the bipartite case of two qubits or qubit-qutrit system, the PPT criterion is also sufficient for separability.

In the multipartite case the situation complicates as one can have many different partitions into set of particles, for example four particle system 1234 can be split, e.g., into 12 - 34 or 1 - 2 - 34. Suppose one splits N particles into p groups, take as an example the split into three groups 1 - 2 - 34. The state is called p -PPT if it has positive *all possible* partial transposes. Fortunately, positivity of partial transpose with respect to certain set of subsystems is the same as positivity with the respect to all remaining subsystems. In the example one should check the positivity of operator obtained after transposition of subsystem 1, next 2, and finally 34.

In what follows, we derive positive partial transpose inequality utilising Bell-Mermin operator [27]. All the p -PPT states were recently shown to satisfy the following inequalities [28]:

$$\text{Tr} [(|\psi^\pm\rangle\langle\psi^\pm| - (1 - 2^{2-p})|\psi^\mp\rangle\langle\psi^\mp|) \rho] \leq 2^{1-p}. \quad (2)$$

If $|\psi^+\rangle$ appears in the first term within the trace, $|\psi^-\rangle$ appears in the second term, and vice versa. Here, $|\psi^\pm\rangle$ is the Greenberger, Horne, and Zeilinger (GHZ) state [29]:

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}} [|0\rangle_1 \dots |0\rangle_N \pm |1\rangle_1 \dots |1\rangle_N]. \quad (3)$$

Omitting the positive factor $2^{2-p} \text{Tr} (|\psi^\mp\rangle\langle\psi^\mp| \rho)$ one arrives at another operator form as follows:

$$\left| \text{Tr} [(|\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-|) \rho] \right| \leq 2^{1-p}. \quad (4)$$

Let us put $p = N$. One has

$$\left| \text{Tr} [(|\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-|) \rho] \right| \leq 2^{1-N}. \quad (5)$$

Of course, every state which has positive partial transposes with respect to all subsystems satisfies the above condition (5). Using the form of the Bell-Mermin operator with two orthogonal settings per site [11]

$$B_N = 2^{(N-1)/2} (|\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-|), \quad (6)$$

the upper bound of the Bell-Mermin inequality, for N -PPT states, is found to read:

$$|\text{Tr}(B_N \rho^{\text{PPT}})| \leq \frac{1}{2^{(N-1)/2}} \quad (7)$$

and it can never reach the local realistic bound for $N \geq 2$. Clearly, the inequality (7) is positive partial transpose inequality. A violation of the inequality implies that the system is in bipartite distillable entanglement since there is at least one bipartite splitting of the system such that the state becomes distillable [19].

We shall show the inequality (7) is stronger than standard Bell inequalities by a factor of $2^{(N-1)/2}$. Suppose the following state

$$\rho_V = V |\psi^+\rangle\langle\psi^+| + (1 - V) \rho_{\text{noise}} \quad (0 \leq V \leq 1). \quad (8)$$

$\rho_{\text{noise}} = \frac{1}{2^N} \mathbb{1}$ is the random noise admixture. The value of V can be interpreted as the reduction factor of the interferometric contrast observed in the multi-particle correlation experiment. It was shown that if the following condition

$$\sum_{i_1, \dots, i_N=1}^2 (\text{Tr} [\rho \sigma_{i_1} \otimes \sigma_{i_2} \otimes \dots \otimes \sigma_{i_N}])^2 \leq 1 \quad (9)$$

is satisfied, all two-setting Bell experiments in the state ρ have local realistic theories [25]. Here, σ_1 and σ_2 are Pauli spin operators satisfying anti-commuting relation $\{\sigma_1, \sigma_2\} = \mathbf{0}$. Especially, the condition (9) is a necessary and sufficient condition for the particular set of states (8) for the existence of local realistic theories for all two-setting Bell experiments. When $V \leq \frac{1}{2^{(N-1)/2}}$, all two-setting Bell experiments in the state ρ_V are reproducible by local realistic theories. In other words, all standard Bell inequalities are satisfied. Assume $V = \frac{1}{2^{(N-1)/2}}$. One has

$$|\text{Tr}(B_N \rho_V)| = 2^{(N-1)/2} V = 1. \quad (10)$$

The positive partial transpose inequality (7) is stronger than the above value by a factor of $2^{(N-1)/2}$. Thus, we have shown that positivity of *every* partial transpose of N -qubit quantum states implies new inequalities on Bell correlations which are stronger than standard Bell inequalities by a factor of $2^{(N-1)/2}$.

In convenience, we introduce so-called ‘positive partial transpose operator’ as

$$P_N = 2^{(N-1)} (|\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-|). \quad (11)$$

Then, the positive partial transpose inequality (7) is expressed as

$$|\langle P_N \rangle| \leq 1. \quad (12)$$

III. MULTIPARTITE BOUND ENTANGLED STATES

In this section, we shall show that a family of Dür bound entangled states violates the positive partial transpose inequality (12) for $N \geq 4$. Such a bound entangled state was introduced as follows by Dür [17]:

$$\rho_N = \frac{1}{N+1} \left(|\psi^+\rangle\langle\psi^+| + \frac{1}{2} \sum_{k=1}^N (P_k + \tilde{P}_k) \right), \quad (13)$$

with P_k being a projector on the state $|0\rangle_1 \dots |1\rangle_k \dots |0\rangle_N$ with “1” on the k th position (\tilde{P}_k is obtained from P_k after replacing “0” by “1” and vice versa). As originally shown in [17] this state violates the Bell-Mermin inequality for $N \geq 8$. It is easy to see that. The Bell-Mermin inequality $|\langle B_N \rangle| \leq 1$ predicts the violation factor of:

$$\text{Tr}[B_N \rho_N] = \frac{2^{(N-1)/2}}{N+1}, \quad (14)$$

which comes from the contribution of the GHZ state $|\psi^+\rangle$ to the bound entangled state by Dür. We find that $\frac{2^{(N-1)/2}}{N+1} > 1$ when $N \geq 8$. Thus, Dür bound entangled states violate the Bell-Mermin inequality for $N \geq 8$. From the argument presented in Ref. [19], a family of Dür bound entangled states has a property that they are bipartite distillable entangled states for $N \geq 8$.

After that, it was shown that Dür bound entangled states violate Bell inequality [20] with three settings per site for $N \geq 7$ [21]. Let us investigate the phenomenon. The Bell operator of the Bell inequality with three settings per site $|\langle B(3)_N \rangle| \leq 1$ is as [11]

$$B(3)_N = \frac{1}{\sqrt{3}} \left(\frac{3}{2} \right)^N (|\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-|). \quad (15)$$

Hence, the Bell inequality predicts the violation factor of:

$$\text{Tr}[B(3)_N \rho_N] = \frac{1}{\sqrt{3}} \left(\frac{3}{2} \right)^N \frac{1}{N+1}, \quad (16)$$

which comes from the contribution of the GHZ state $|\psi^+\rangle$ to the bound entangled state by Dür. We find that $\frac{1}{\sqrt{3}} \left(\frac{3}{2} \right)^N \frac{1}{N+1} > 1$ when $N \geq 7$. Thus, Dür bound entangled states violate the Bell inequality for $N \geq 7$. From similar to the argument presented in Ref. [19], a family of Dür bound entangled states has a property that they are bipartite distillable entangled states for $N \geq 7$.

It also turned out that Dür bound entangled states violate Bell inequality [22] with a continuous range of settings of the apparatus at each site for $N \geq 6$ [23]. Let us investigate the phenomenon. The Bell operator of the

Bell inequality with a continuous range of settings of the apparatus at each site $|\langle B(\infty)_N \rangle| \leq 1$ is as [11, 30]

$$B(\infty)_N = \frac{1}{2} \left(\frac{\pi}{2} \right)^N (|\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-|). \quad (17)$$

Hence, the Bell inequality predicts the violation factor of:

$$\text{Tr}[B(\infty)_N \rho_N] = \frac{1}{2} \left(\frac{\pi}{2} \right)^N \frac{1}{N+1}, \quad (18)$$

which comes from the contribution of the GHZ state $|\psi^+\rangle$ to the bound entangled state by Dür. We find that $\frac{1}{2} \left(\frac{\pi}{2} \right)^N \frac{1}{N+1} > 1$ when $N \geq 6$. We see Dür bound entangled states violate the Bell inequality for $N \geq 6$. From similar to the argument presented in Ref. [19], a family of Dür bound entangled states has a property that they are bipartite distillable entangled states for $N \geq 6$.

The positive partial transpose inequality (12) predicts the substantially bigger violation factor of:

$$\text{Tr}[P_N \rho_N] = \frac{2^{(N-1)}}{N+1}, \quad (19)$$

which comes from the contribution of the GHZ state $|\psi^+\rangle$ to the bound entangled state by Dür. We find that $\frac{2^{(N-1)}}{N+1} > 1$ when $N \geq 4$. Thus, Dür bound entangled states violate the new inequality for $N \geq 4$. From similar to the argument presented in Ref. [19], a family of Dür bound entangled states has a property that they are bipartite distillable entangled states for $N \geq 4$.

IV. SUMMARY

In summary, we have shown that positivity of *every* partial transpose of N -qubit quantum states implies new inequalities on Bell correlations which are stronger than standard Bell inequalities by a factor of $2^{(N-1)/2}$. A violation of the inequality implies that the system is in bipartite distillable entanglement since there is at least one bipartite splitting of the system such that the state becomes distillable. It turned out that Dür bound entangled states violate the new inequality for $N \geq 4$. Thus, a family of Dür bound entangled states has a property that they are bipartite distillable entangled states for $N \geq 4$.

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